CS310

P vs NP

the steel cage death match

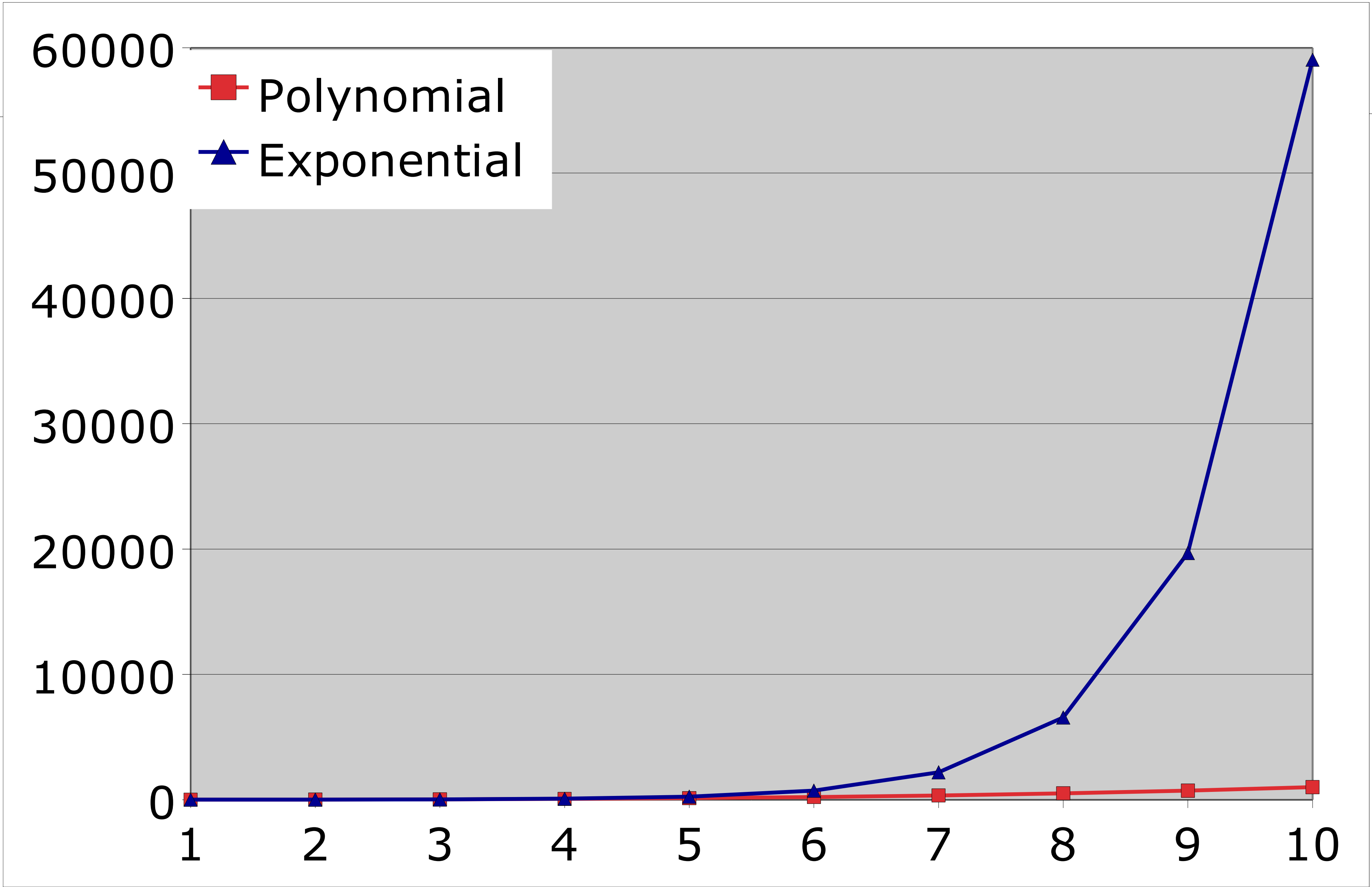
How hard is a problem to solve?

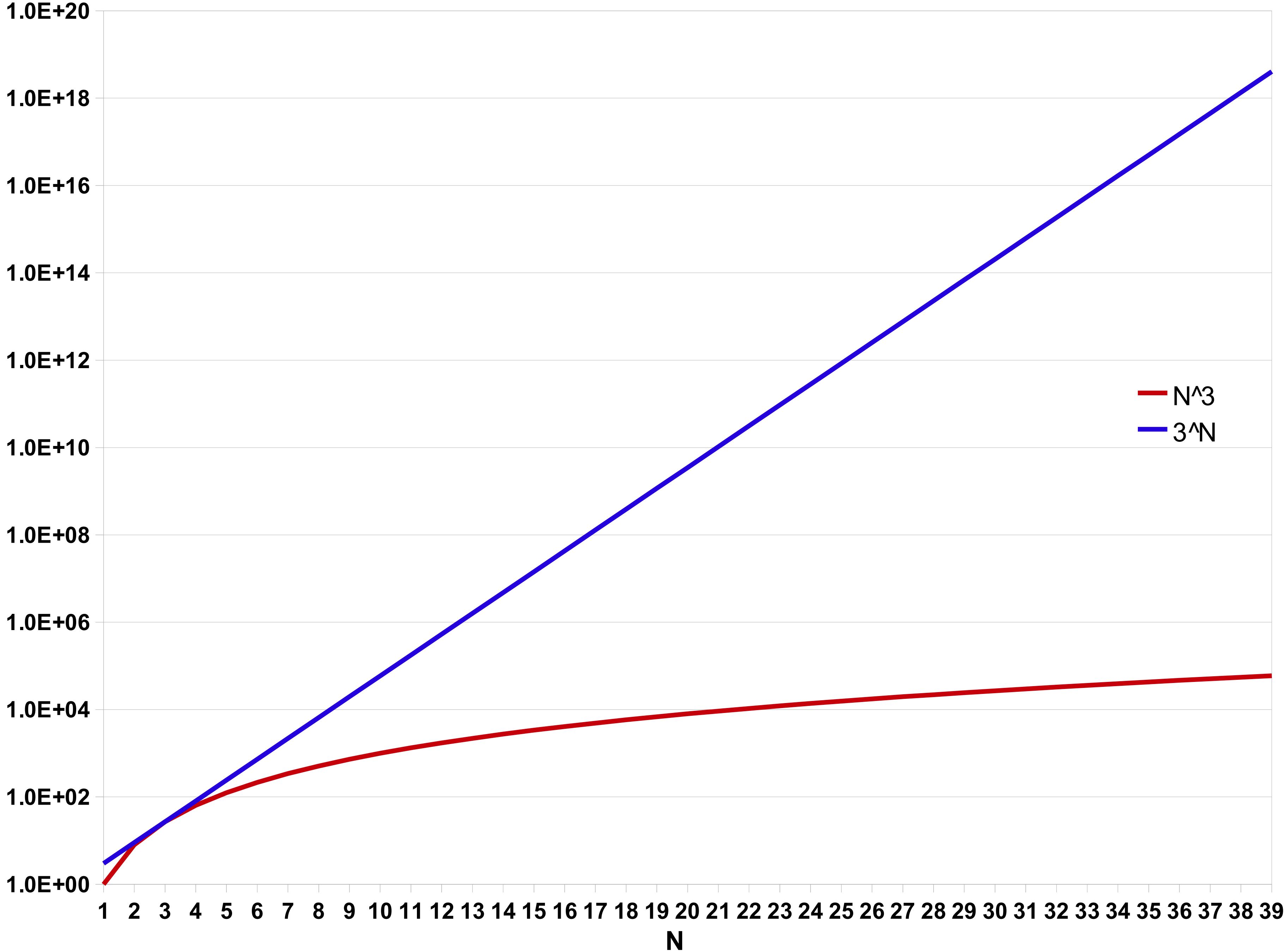
Section 7.2

December 1, 2010

Polynomial vs Exponential

- Polynomial: n^3 Exponential: 3^n




1.0E+20
1.0E+18
1.0E+16
1.0E+14
1.0E+12
1.0E+10
1.0E+08
1.0E+06
1.0E+04
1.0E+02
1.0E+00
1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34 35 36 37 38 39
N
N^3
3^N

Complexity relationships between models

- Theorem 7.8: let t(n) >= n, every t(n) time multi-tape TM has an equivalent $O((t(n)^2)$ time single-tape TM.
  - polynomial difference
- Theorem 7.9: Every t(n) >= n time ND single tape TM has an equivalent $2^{O(t(n))}$ time deterministic single tape TM
  - exponential difference

The class P

- P is the class of languages that are decidable in polynomial time on a deterministic, single tape TM
- Problems in class P
  - PATH: { <G,s,d> | G is a directed graph, find a directed path from s to d }
  - RELPRIME: {<x, y> | x and y are relatively prime}
    - Euclidean algorithm
- Every context-free language is in P

RELPRIME Sipser, p 261

PROOF The Euclidean algorithm E is as follows.

$E =$ "On input $\langle x, y \rangle$, where x and y are natural numbers in binary:

1. Repeat until $y = 0$:
2. Assign $x \leftarrow x \bmod y$.
3. Exchange x and y.
4. Output x."

Algorithm R solves *RELPRIME*, using E as a subroutine.

$R =$ "On input $\langle x, y \rangle$, where x and y are natural numbers in binary:

1. Run E on $\langle x, y \rangle$.
2. If the result is 1, *accept*. Otherwise, *reject*."

CFG Parsing   Sipser p 263

$D =$ "On input $w = w_1 \cdots w_n$:

1. If $w = \varepsilon$ and $S \to \varepsilon$ is a rule, *accept*. ⟦ handle $w = \varepsilon$ case ⟧
2. For $i = 1$ to n: ⟦ examine each substring of length 1 ⟧
3. For each variable A:
4. Test whether $A \to \mathtt{b}$ is a rule, where $\mathtt{b} = w_i$.
5. If so, place A in $table(i, i)$.
6. For $l = 2$ to n: ⟦ l is the length of the substring ⟧
7. For $i = 1$ to $n - l + 1$: ⟦ i is the start position of the substring ⟧
8. Let $j = i + l - 1$, ⟦ j is the end position of the substring ⟧
9. For $k = i$ to $j - 1$: ⟦ k is the split position ⟧
10. For each rule $A \to BC$:
11. If $table(i, k)$ contains B and $table(k + 1, j)$ contains C, put A in $table(i, j)$.
12. If S is in $table(1, n)$, *accept*. Otherwise, *reject*."

Real Life

- Problems in class P are usually manageable on a real computer
  - n^K
  - though k=100 may introduce some practical problems

The class NP

- NP is the class of languages that are decidable in polynomial time on a nondeterministic single tape TM
  - Problems in class NP
    - HAMPATH: { <G, s, t> | G is a directed graph, with Hamilton path from s to t } (a path that passes through every vertex of a graph exactly once)
    - CLIQUE : {<G,k> | G is an undirected graph with a k-clique}
      - K-clique a subgraph wherein every two nodes are connected by an edge.
  - These problems are decidable on a deterministic single tape TM in exponential time

Verifier

- A verifier of a language, A, is an algorithm, V, such that

  A = { w | V accepts <w, c> for some string c}

  where c is a certificate

  |c| is polynomial in terms of |w|
- NP is the class of languages that have polynomial time (in terms of the length of w) verifiers

Clique, Sipser p 268

PROOF The following is a verifier V for *CLIQUE*.

V = "On input $\langle\langle G, k\rangle, c\rangle$:

1. Test whether c is a set of k nodes in G
2. Test whether G contains all edges connecting nodes in c.
3. If both pass, *accept*; otherwise, *reject*."

ALTERNATIVE PROOF If you prefer to think of NP in terms of nondeterministic polynomial time Turing machines, you may prove this theorem by giving one that decides *CLIQUE*. Observe the similarity between the two proofs.

N = "On input $\langle G, k\rangle$, where G is a graph:

1. Nondeterministically select a subset c of k nodes of G.
2. Test whether G contains all edges connecting nodes in c.
3. If yes, *accept*; otherwise, *reject*."

Subset-Sum Sipser p 269

PROOF The following is a verifier V for *SUBSET-SUM*.

$V =$ "On input $\langle\langle S, t\rangle, c\rangle$:

1. Test whether c is a collection of numbers that sum to t.
2. Test whether S contains all the numbers in c.
3. If both pass, *accept*; otherwise, *reject*."

ALTERNATIVE PROOF We can also prove this theorem by giving a nondeterministic polynomial time Turing machine for *SUBSET-SUM* as follows.

$N =$ "On input $\langle S, t\rangle$:

1. Nondeterministically select a subset c of the numbers in S.
2. Test whether c is a collection of numbers that sum to t.
3. If the test passes, *accept*; otherwise, *reject*."

P vs NP

- $P \subseteq NP$
  - unknown if the classes are unequal
- If P = NP, then all problems in NP can be solved in polynomial time, if we are clever enough to find the right algorithm

NP-Complete

- NP-Completeness
  - set of problems in NP whose complexity is related to all problems in NP
  - if an NP-Complete problem can be shown to be in P, then P=NP
  - boolean satisfiability, for example
  - vertex-cover
  - clique
  - Hamilton Path

Recent Work

- Claim by Vinay Deolalikar (from HP Labs) that N != NP

- https://rjlipton.wordpress.com/2010/08/08/a-proof-that-p-is-not-equal-to-np/
  - Link to Deolalikar's paper
  - Much discussion

- https://rjlipton.wordpress.com/2010/08/12/fatal-flaws-in-deolalikars-proof/
  - Fatal flaws?