3. Functions
3.1 • What Is a Function?
What is a Function?

Perhaps the most useful mathematical idea for modeling the real world is the concept of function, which we study in this chapter.

- In this section, we explore the idea of a function and then give the mathematical definition of function.
Functions All Around Us
Functions All Around Us

• For example,
  ◦ Your height depends on your age.
  ◦ The temperature depends on the date.
  ◦ The cost of mailing a package depends on its weight.
We use the term function to describe this dependence of one quantity on another.

That is, we say:

- Height is a function of age.
- Temperature is a function of date.
- Cost of mailing a package is a function of weight.
Functions All Around Us

Here are some more examples:

- The area of a circle is a function of its radius.
- The number of bacteria in a culture is a function of time.
- The weight of an astronaut is a function of her elevation.
- The price of a commodity is a function of the demand for that commodity.
The rule that describes how the area $A$ of a circle depends on its radius $r$ is given by the formula

$$A = \pi r^2$$
Even when a precise rule or formula describing a function is not available, we can still describe the function by a graph.

- For example, when you turn on a hot water faucet, the temperature of the water depends on how long the water has been running.
- So we can say that: Temperature of water from the faucet is a function of time.
The figure shows a rough graph of the temperature $T$ of the water as a function of the time $t$ that has elapsed since the faucet was turned on.
Temperature of Water from a Faucet

- The graph shows that the initial temperature of the water is close to room temperature.
- When the water from the hot water tank reaches the faucet, the water’s temperature $T$ increases quickly.
Temperature of Water from a Faucet

- In the next phase, $T$ is constant at the temperature of the water in the tank.
  - When the tank is drained, $T$ decreases to the temperature of the cold water supply.
Function

• A function is a rule.

  ◦ To talk about a function, we need to give it a name.

  ◦ We will use letters such as $f, g, h, \ldots$ to represent functions.

  ◦ For example, we can use the letter $f$ to represent a rule as follows:
    “$f$” is the rule “square the number”
When we write $f(2)$, we mean “apply the rule $f$ to the number 2.”

- Applying the rule gives $f(2) = 2^2 = 4$.

- Similarly,
  
  $f(3) = 3^2 = 9$
  
  $f(4) = 4^2 = 16$

  and, in general,

  $f(x) = x^2$
A function $f$ is:

- A rule that assigns to each element $x$ in a set $A$ exactly one element, called $f(x)$, in a set $B$. 
Domain & Range

- The set $A$ is called the domain of the function.

- The range of $f$ is the set of all possible values of $f(x)$ as $x$ varies throughout the domain, that is,

$$\text{range of } f = \{f(x) \mid x \in A\}$$
Independent and Dependent Variables

- The symbol that represents an arbitrary number in the domain of a function \( f \) is called an independent variable.

- The symbol that represents a number in the range of \( f \) is called a dependent variable.
So, if we write

\[ y = f(x) \]

then

- \( x \) is the independent variable.
- \( y \) is the dependent variable.
It’s helpful to think of a function as a machine, actually think about it as a computer.

If \( x \) is in the domain of the function \( f \), then when \( x \) enters the machine, it is accepted as an input and the machine produces an output \( f(x) \), according to the rule of the function.

**Figure 3**
The domain as the set of all possible inputs.

The range as the set of all possible outputs.
Another way to picture a function is by an arrow diagram.

- Each arrow connects an element of $A$ to an element of $B$.
- The arrow indicates that $f(x)$ is associated with $x$, $f(a)$ is associated with $a$, and so on.
A function $f$ is defined by the formula $f(x) = x^2 + 4$

(a) Express in words how $f$ acts on the input $x$ to produce the output $f(x)$.

(b) Evaluate $f(3)$, $f(-2)$, and $f(\sqrt{5})$.

(c) Find the domain and range of $f$.

(d) Draw a machine diagram for $f$. 
E.g. 1—The Squaring Function

The formula tells us that \( f \) first squares the input \( x \) and then adds 4 to the result.

- So \( f \) is the function “square, then add 4”
The values of $f$ are found by substituting for $x$ in $f(x) = x^2 + 4$.

- $f(3) = 3^2 + 4 = 13$
- $f(-2) = (-2)^2 + 4 = 8$
- $f(\sqrt{5}) = (\sqrt{5})^2 + 4 = 9$
The domain of $f$ consists of all possible inputs for $f$.

- Since we can evaluate the formula for every real number $x$, the domain of $f$ is the set $\mathbb{R}$ of all real numbers.

The range of $f$ consists of all possible outputs of $f$.

- Since $x^2 \geq 0$ for all real numbers $x$, we have $x^2 + 4 \geq 4$, so for every output of $f$ we have $f(x) \geq 4$.

Thus, the range of $f$ is:

$$\{y \mid y \geq 4\} = [4, \infty)$$
Here’s a machine diagram for the function.

![Machine Diagram]

**FIGURE 5** Machine diagram
Evaluating a Function

- In the definition of a function the independent variable $x$ plays the role of a “placeholder.”

  - For example, the function $f(x) = 3x^2 + x - 5$ can be thought of as:
    
    $f(__) = 3 \cdot __^2 + __ - 5$

  - To evaluate $f$ at a number, we substitute the number for the placeholder.
E.g. 2—Evaluating a Function

Let \( f(x) = 3x^2 + x - 5 \).

Evaluate each function value.

(a) \( f(-2) \)
(b) \( f(0) \)
(c) \( f(4) \)
(d) \( f(\frac{1}{2}) \)
Evaluating a Function

• To evaluate $f$ at a number, we substitute the number for $x$ in the definition of $f$.

(a) $f(-2) = 3 \cdot (-2)^2 + (-2) - 5 = 5$

(b) $f(0) = 3 \cdot 0^2 + 0 - 5 = -5$

(c) $f(4) = 3 \cdot 4^2 + 4 - 5 = 47$

(d) $f \left( \frac{1}{2} \right) = 3 \cdot \left( \frac{1}{2} \right)^2 + \frac{1}{2} - 5 = -\frac{15}{4}$
A piecewise defined function

- A cell phone plan costs $39 a month.

  - The plan includes 400 free minutes and charges 20¢ for each additional minute of usage.
  - The monthly charges are a function of the number of minutes used, given by:

\[
C(x) = \begin{cases} 
39 & \text{if } 0 \leq x \leq 400 \\
39 + 0.20(x - 400) & \text{if } x > 400 
\end{cases}
\]

- Find \(C(100), C(400),\) and \(C(480).\)
A Piecewise Defined Function

- Remember that a function is a rule.

- Here’s how we apply the rule for this function.
  - First, we look at the value of the input $x$.
  - If $0 \leq x \leq 400$, then the value of $C(x)$ is: 39
  - However, if $x > 400$, then the value of $C(x)$ is:
    - $39 + 0.2(x - 400)$
Since $100 \leq 400$, we have $C(100) = 39$.

Since $400 \leq 400$, we have $C(400) = 39$.

Since $480 > 400$, we have $C(480) = 39 + 0.2(480 - 400) = 55$.

Thus, the plan charges: $39$ for $100$ minutes, $39$ for $400$ minutes, and $55$ for $480$ minutes.
The Weight of an Astronaut

- If an astronaut weighs 130 pounds on the surface of the earth, then her weight when she is $h$ miles above the earth is given by the function

$$w(h) = 130 \left( \frac{3960}{3960 + h} \right)^2$$
The Weight of an Astronaut

- (a) What is her weight when she is 100 mi above the earth?

- (b) Construct a table of values for the function \( w \) that gives her weight at heights from 0 to 500 mi.
  
  - What do you conclude from the table?
E.g. 5—The Weight of an Astronaut

- We want the value of the function $w$ when $h = 100$.
- That is, we must calculate $w(100)$.

$$w(100) = 130 \left( \frac{3960}{3960 + 100} \right)^2 \approx 123.67$$

- So, at a height of 100 mi, she weighs about 124 lb.
The Weight of an Astronaut

- The table gives the astronaut’s weight, rounded to the nearest pound, at 100-mile increments.
  - The values are calculated as in part (a).
  - The table indicates that, the higher the astronaut travels, the less she weighs.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$w(h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>130</td>
</tr>
<tr>
<td>100</td>
<td>124</td>
</tr>
<tr>
<td>200</td>
<td>118</td>
</tr>
<tr>
<td>300</td>
<td>112</td>
</tr>
<tr>
<td>400</td>
<td>107</td>
</tr>
<tr>
<td>500</td>
<td>102</td>
</tr>
</tbody>
</table>
The Weight of an Astronaut

- The table is the precursor to visualizing the data as a graph.
- The graph will help you see the valid values of the function.
- For example, this shows you that the weight is inversely proportional to the altitude, the higher you go, the less you weigh.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$w(h)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>130</td>
</tr>
<tr>
<td>100</td>
<td>124</td>
</tr>
<tr>
<td>200</td>
<td>118</td>
</tr>
<tr>
<td>300</td>
<td>112</td>
</tr>
<tr>
<td>400</td>
<td>107</td>
</tr>
<tr>
<td>500</td>
<td>102</td>
</tr>
</tbody>
</table>
The Domain of a Function

THINK INPUT TO THE COMPUTER (THE FUNCTION MACHINE)
Recall that the domain of a function is the set of all inputs for the function.

- The domain of a function may be stated explicitly.

- For example, if we write
  - \( f(x) = x^2, \quad 0 \leq x \leq 5 \)
  - then the domain is the set of all real numbers \( x \) for which \( 0 \leq x \leq 5 \).
Domain of a Function

- If the function is given by an algebraic expression and the domain is not stated explicitly, then, by convention, the domain of the function is:
  - The domain of the algebraic expression—that is, the set of all real numbers for which the expression is defined as a real number.
Domain of a Function

For example, consider the functions

\[ f(x) = \frac{1}{x - 4} \quad g(x) = \sqrt{x} \]

- The function \( f \) is not defined at \( x = 4 \). So, its domain is \( \{x \mid x \neq 4\} \).

- The function \( g \) is not defined for negative \( x \). So, its domain is \( \{x \mid x \neq 0\} \).
Finding Domains of Functions

- Find the domain of each function.

(a) \( f(x) = \frac{1}{x^2 - x} \)

(b) \( g(x) = \sqrt{9 - x^2} \)

(c) \( h(t) = \frac{t}{\sqrt{t + 1}} \)
Finding Domains

- The function is not defined when the denominator is 0.

  ◦ Since

  \[ f(x) = \frac{1}{x^2 - x} = \frac{1}{x(x - 1)} \]

  we see that \( f(x) \) is not defined when \( x = 0 \) or \( x = 1 \).
Finding Domains

- Thus, the domain of \( f \) is:
  \[
  \{ x \mid x \neq 0, x \neq 1 \}
  \]

- The domain may also be written in interval notation as:
  \[
  (\infty, 0) \ 
  (0, 1) \ 
  (1, \infty)
  \]
Finding Domains

- We can’t take the square root of a negative number.
- So, we must have $9 - x^2 \geq 0$.

  - Using the methods of Section 1.6, we can solve this inequality to find that:
    - $-3 \leq x \leq 3$
  - Thus, the domain of $g$ is:
    - $\{x \mid -3 \leq x \leq 3\} = [-3, 3]$
Finding Domains

- We can’t take the square root of a negative number, and we can’t divide by 0.

- So, we must have \( t + 1 > 0 \), that is, \( t > -1 \).

  - Thus, the domain of \( h \) is:

    \[
    \{ t \mid t > -1 \} = (-1, \infty)
    \]
• Four Ways to Represent a Function
Four Ways to Represent a Function

To help us understand what a function is, we have used:

- Machine diagram
- Arrow diagram
We can describe a specific function in these ways:

- Verbally (a description in words)
- Algebraically (an explicit formula)
- Visually (a graph)
- Numerically (a table of values)
A single function may be represented in all four ways.

- It is often useful to go from one representation to another to gain insight into the function.
- However, certain functions are described more naturally by one method than by the others.
Verbal Representation

- An example of a verbal description is the following rule for converting between temperature scales:
  - “To find the Fahrenheit equivalent of a Celsius temperature, multiply the Celsius temperature by 9/5, then add 32.”
  - In Example 7, we see how to describe this verbal rule or function algebraically, graphically, and numerically.
A useful representation of the area of a circle as a function of its radius is the algebraic formula

\[ A(r) = \pi r^2 \]
Visual Representation

- The graph produced by a seismograph is a visual representation of the vertical acceleration function $a(t)$ of the ground during an earthquake.
Verbal Representation

Finally, consider the function $C(w)$.

- It is described verbally as:

  “the cost of mailing a first-class letter with weight $w$."

The most convenient way of describing this function is numerically—using a table of values.

<table>
<thead>
<tr>
<th>$w$ (ounces)</th>
<th>$C(w)$ (dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 &lt; w \leq 1$</td>
<td>1.13</td>
</tr>
<tr>
<td>$1 &lt; w \leq 2$</td>
<td>1.30</td>
</tr>
<tr>
<td>$2 &lt; w \leq 3$</td>
<td>1.47</td>
</tr>
<tr>
<td>$3 &lt; w \leq 4$</td>
<td>1.64</td>
</tr>
<tr>
<td>$4 &lt; w \leq 5$</td>
<td>1.81</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
</tbody>
</table>
Representing a Function in Four Ways

- Let $F(C)$ be the Fahrenheit temperature corresponding to the Celsius temperature $C$.
  - Thus, $F$ is the function that converts Celsius inputs to Fahrenheit outputs.
  - We have already seen the verbal description of this function.
Representing a Function in Four Ways

- Find way to represent this function
- Algebraically (using a formula)
- Numerically (using a table of values)
- Visually (using a graph)
Representing a Function

- The verbal description tells us that we first multiply the input $C$ by $\frac{9}{5}$ and then add 32 to the result.

- So we get

$$F(C) = \frac{9}{5}C + 32$$
Representing a Function

- We use the algebraic formula for F that we found in part (a) to construct a table of values.

<table>
<thead>
<tr>
<th>$C$ (Celsius)</th>
<th>$F$ (Fahrenheit)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-10$</td>
<td>14</td>
</tr>
<tr>
<td>0</td>
<td>32</td>
</tr>
<tr>
<td>10</td>
<td>50</td>
</tr>
<tr>
<td>20</td>
<td>68</td>
</tr>
<tr>
<td>30</td>
<td>86</td>
</tr>
<tr>
<td>40</td>
<td>104</td>
</tr>
</tbody>
</table>
Representing a Function

- We use the points tabulated in part (b) to help us draw the graph of this function.