1.6 Inequalities

Inequalities

- Some problems in algebra lead to inequalities instead of equations.

- An inequality looks just like an equation—except that, in the place of the equal sign
is one of these symbols: $<$, $>$, $\leq$, or $\geq$.

  - Here is an example: $4x + 7 \leq 19$

Inequalities

- The table shows that some numbers satisfy the inequality and some numbers don't.

x	$4x + 7 \le 19$
1	$11 \le 19$ ✓
2	$15 \le 19$ ✓
3	$19 \le 19$ ✓
4	$23 \le 19$ ✗
5	$27 \le 19$ ✗

Solving Inequalities

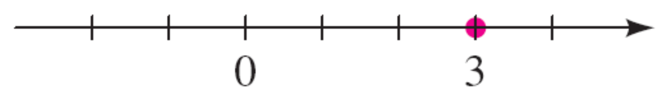
- To solve an inequality that contains a variable means to find all values of the variable that make the inequality true.

  - Unlike an equation, an inequality generally has infinitely many solutions.

  - These form an interval or a union of intervals on the real line.

Solving Inequalities

- The following illustration shows how an inequality differs from its corresponding equation:

		Solution	Graph
Equation:	$4x + 7 = 19$	$x = 3$	
Inequality:	$4x + 7 \leq 19$	$x \leq 3$	

Solving Inequalities

- To solve inequalities, we use the following rules to isolate the variable on one side of the inequality sign.

  - These rules tell us when two inequalities are equivalent ($\Leftrightarrow$ means "is equivalent to").

  - In these rules, the symbols *A*, *B*, and *C* stand for real numbers or algebraic expressions.

Solving Inequalities

- Here, we state the rules for inequalities involving the symbol ≤.
  - However, they apply to all four inequality symbols.

Rules for Inequalities

Rule	Description
1. $A \le B \quad \Leftrightarrow \quad A + C \le B + C$	**Adding** the same quantity to each side of an inequality gives an equivalent inequality.
2. $A \le B \quad \Leftrightarrow \quad A - C \le B - C$	**Subtracting** the same quantity from each side of an inequality gives an equivalent inequality.
3. If $C > 0$, then $A \le B \quad \Leftrightarrow \quad CA \le CB$	**Multiplying** each side of an inequality by the same *positive* quantity gives an equivalent inequality.
4. If $C < 0$, then $A \le B \quad \Leftrightarrow \quad CA \ge CB$	**Multiplying** each side of an inequality by the same *negative* quantity *reverses the direction* of the inequality.
5. If $A > 0$ and $B > 0$, then $A \le B \quad \Leftrightarrow \quad \dfrac{1}{A} \ge \dfrac{1}{B}$	**Taking reciprocals** of each side of an inequality involving *positive* quantities *reverses the direction* of the inequality.
6. If $A \le B$ and $C \le D$, then $A + C \le B + D$	Inequalities can be added.

Solving Inequalities

- Pay special attention to Rules 3 and 4.

  - Rule 3 says that we can multiply (or divide) each side of an inequality by a positive number.

  - However, Rule 4 says that, if we multiply each side of an inequality by a negative number, then we reverse the direction of the inequality.

Solving Inequalities

- For example, if we start with the inequality
  - $3 < 5$
- and multiply by 2, we get:
  - $6 < 10$

- However, if we multiply by –2, we get:

$$-6 > -10$$

Solving Linear Inequalities

Linear Inequalities

- An inequality is linear if:
  - Each term is constant or a multiple of the variable.
  - To solve a linear inequality, we isolate the variable on one side of the inequality sign.

E.g. 1—Solving a Linear Inequality

- Solve the inequality

$$3x < 9x + 4$$

and sketch the solution set.

E.g. 1—Solving a Linear Inequality

- $3x < 9x + 4$
- $3x - 9x < 9x + 4 - 9x$ (Subtract 9x)
- $-6x < 4$ (Simplify)
- $(-1/6)(-6x) > (-1/6)(4)$ (Multiply by –1/6 or divide by –6)
- $x > -2/3$ (Simplify)

  - The solution set consists of all numbers greater than –2/3.

E.g. 1—Solving a Linear Inequality

- In other words, the solution of the inequality is the interval (–2/3, ∞).

Figure 1

E.g. 2—Solving a Pair of Simultaneous Inequalities

- Solve the inequalities

$$4 \le 3x - 2 < 13$$

  - The solution set consists of all values of x that satisfy both of the inequalities

$$4 \le 3x - 2 \quad \text{and} \quad 3x - 2 < 13$$

E.g. 2—Solving a Pair of Simultaneous Inequalities

- Using Rules 1 and 3, we see that these inequalities are equivalent:
  - $4 \le 3x - 2 < 13$
  - $6 \le 3x < 15$ (Add 2)
  - $2 \le x < 5$ (Divide by 3)

E.g. 2—Solving a Pair of Simultaneous Inequalities

- Therefore, the solution set is

$$[2, 5)$$

Figure 2

Solving Nonlinear Inequalities

Nonlinear Inequalities

- To solve inequalities involving squares and other powers, we use factoring, together with the following principle.

The Sign of a Product or Quotient

- If a product or a quotient has an even number of negative factors, then its value is positive.

- If a product or a quotient has an odd number of negative factors, then its value is negative.

Solving Nonlinear Inequalities

- For example, to solve the inequality

$$x^2 - 5x \leq -6$$

- We first move all the terms to the left-hand side and factor to get

$$(x - 2)(x - 3) \leq 0$$

  - This form of the inequality says that the product $(x - 2)(x - 3)$ must be negative or zero.

Solving Nonlinear Inequalities

- So to solve the inequality, we must determine where each factor is negative or positive.
  - This is because the sign of a product depends on the sign of the factors.
  - The details are explained in Example 3, in which we use the following guidelines.

Guidelines for Solving Nonlinear Inequalities

- Example 3 illustrates the following guidelines for solving an inequality that can be factored.

1. Move all terms to one side.
2. Factor.
3. Find the intervals.
4. Make a table or diagram.
5. Solve.

Guideline 1 for Solving Nonlinear Inequalities

- **Move all terms to one side.**
  - If necessary, rewrite the inequality so that all nonzero terms appear on one side of the inequality sign.
  - If the nonzero side of the inequality involves quotients, bring them to a common denominator.

Guideline 2 for Solving Nonlinear Inequalities

- **Factor.**
  - Factor the nonzero side of the inequality.

Guideline 3 for Solving Nonlinear Inequalities

- Find the intervals.
  - Determine the values for which each factor is zero.
  - These numbers will divide the real line into intervals.
  - List the intervals determined by these numbers.

Guideline 4 for Solving Nonlinear Inequalities

- Make a table or diagram.
  - Use test values to make a table or diagram of the signs of each factor on each interval.
  - In the last row of the table, determine the sign of the product (or quotient) of these factors.

Guideline 5 for Solving Nonlinear Inequalities

Solve.

- Determine the solution of the inequality from the last row of the sign table.
- Be sure to check whether the inequality is satisfied by some or all of the endpoints of the intervals.
- This may happen if the inequality involves $\leq$ or $\geq$.

Guidelines for Solving Nonlinear Inequalities

- The factoring technique described in these guidelines works only if all nonzero terms appear on one side of the inequality symbol.

  - If the inequality is not written in this form, first rewrite it—as indicated in Step 1.

E.g. 3—A Quadratic Inequality

- Solve the inequality
  - $x^2 \leq 5x - 6$

- First, we move all the terms to the left-hand side

$$x^2 - 5x + 6 \leq 0$$

- Factoring the left side of the inequality, we get

$$(x - 2)(x - 3) \leq 0$$

E.g. 3—A Quadratic Inequality

- The factors of the left-hand side are $x - 2$ and $x - 3$.
  - These factors are zero when x is 2 and 3, respectively.

E.g. 3—A Quadratic Inequality

- As shown, the numbers 2 and 3 divide the real line into three intervals:

$$(-\infty, 2), (2, 3), (3, \infty)$$

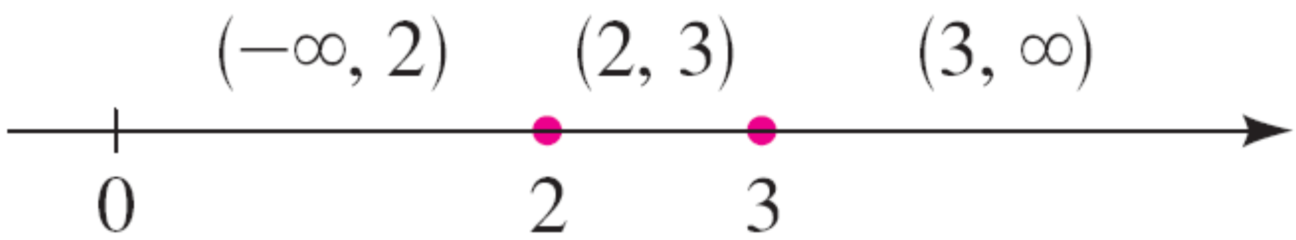


Figure 3

E.g. 3—A Quadratic Inequality

- The factors $x - 2$ and $x - 3$ change sign only at 2 and 3, respectively.
  - So, they maintain their signs over the length of each interval.

E.g. 3—A Quadratic Inequality

- On each interval, we determine the signs of the factors using test values.
  - We choose a number inside each interval and check the sign of the factors $x - 2$ and $x - 3$ at the value selected.

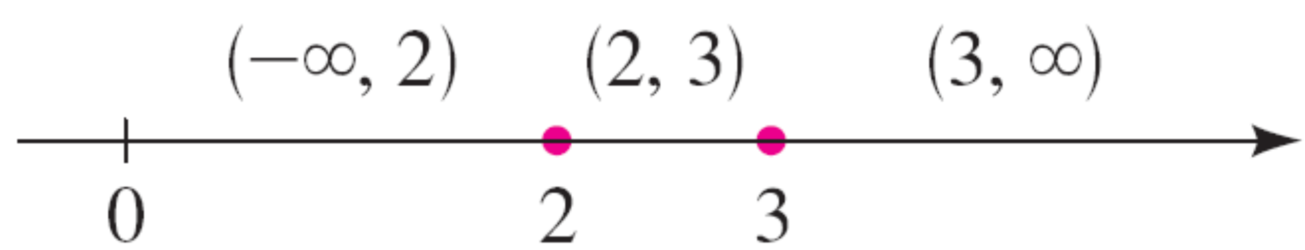


Figure 3

E.g. 3—A Quadratic Inequality

- For instance, let's use the test value $x = 1$ for the interval $(-\infty, 2)$.

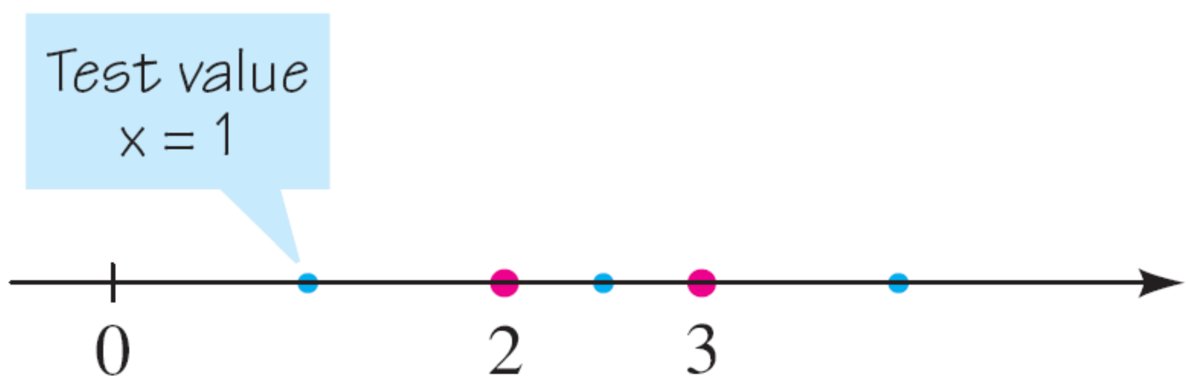


Figure 4

E.g. 3—A Quadratic Inequality

- Then, substitution in the factors $x - 2$ and $x - 3$ gives:
  - $x - 2 = 1 - 2 = -1 < 0$
- and
  - $x - 3 = 1 - 3 = -2 < 0$
  - So, both factors are negative on this interval.
  - Notice that we need to check only one test value for each interval because the factors $x - 2$ and $x - 3$ do not change sign on any of the three intervals we found.

E.g. 3—A Quadratic Inequality

- Using the test values $x = 2\frac{1}{2}$ and $x = 4$ for the intervals (2, 3) and (3, ∞), respectively, we construct the following sign table.

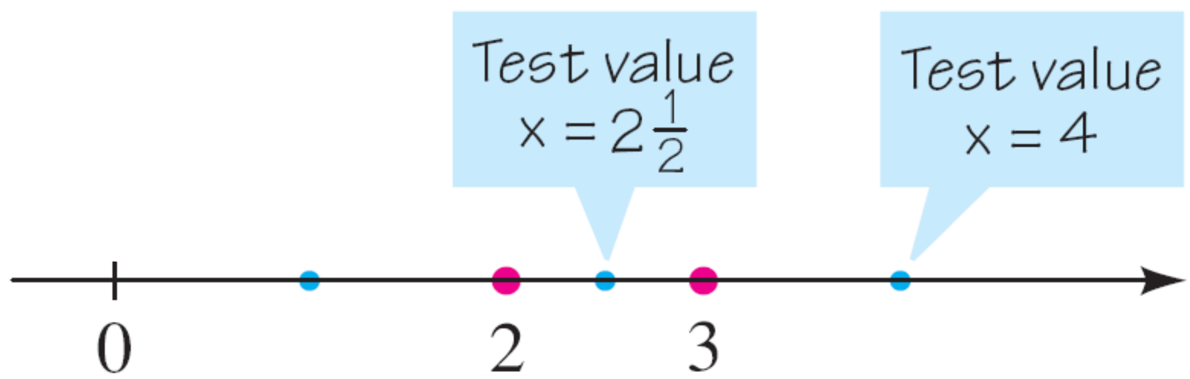


Figure 4

E.g. 3—A Quadratic Inequality

- The final row is obtained from the fact that the expression in the last row is the product of the two factors.

Interval	$(-\infty, 2)$	$(2, 3)$	$(3, \infty)$
Sign of $x - 2$	$-$	$+$	$+$
Sign of $x - 3$	$-$	$-$	$+$
Sign of $(x - 2)(x - 3)$	$+$	$-$	$+$

E.g. 3—A Quadratic Inequality

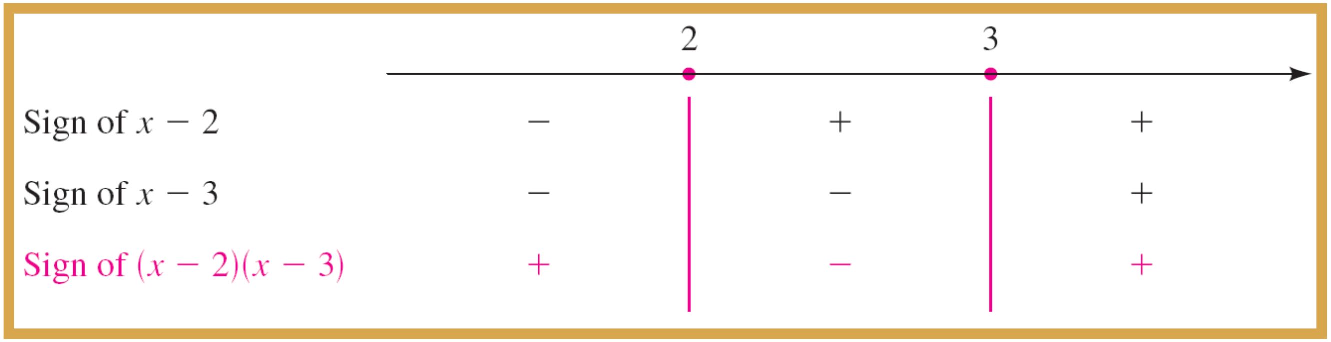
- If you prefer, you can represent that information on a real number line—as in this sign diagram.

  - The vertical lines indicate the points at which the real line is divided into intervals.

		2		3	
Sign of $x - 2$	$-$		$+$		$+$
Sign of $x - 3$	$-$		$-$		$+$
Sign of $(x - 2)(x - 3)$	$+$		$-$		$+$

E.g. 3—A Quadratic Inequality

- We read from the table or the diagram that $(x - 2)(x - 3)$ is negative on the interval $(2, 3)$.



Interval	$(-\infty, 2)$	$(2, 3)$	$(3, \infty)$
Sign of $x - 2$	−	+	+
Sign of $x - 3$	−	−	+
Sign of $(x - 2)(x - 3)$	+	−	+

E.g. 3—A Quadratic Inequality

- Thus, the solution of the inequality $(x - 2)(x - 3) \leq 0$ is:

$$\{x \mid 2 \leq x \leq 3\} = [2, 3]$$

  - We have included the endpoints 2 and 3 because we seek values of x such that the product is either less than or equal to zero.

E.g. 3—A Quadratic Inequality

- The solution is illustrated here.

Figure 5

E.g. 4—Solving an Inequality

- Solve the inequality
  - $2x^2 - x > 1$

- First, we move all the terms to the left-hand side

$$2x^2 - x - 1 > 0$$

- Factoring the left side of the inequality, we get

$$(2x + 1)(x - 1) > 0$$

E.g. 4—Solving an Inequality

- The factors of the left-hand side are $2x + 1$ and $x - 1$.
  - These factors are zero when x is $-1/2$ and 1, respectively.
  - These numbers divide the real line into the intervals

$$(-\infty, -1/2), (-1/2, 1), (1, \infty)$$

E.g. 4—Solving an Inequality

- We make the following diagram, using test points to determine the sign of each factor in each interval.

	$(-\infty, -\frac{1}{2})$	$(-\frac{1}{2}, 1)$	$(1, \infty)$
Sign of $2x + 1$	$-$	$+$	$+$
Sign of $x - 1$	$-$	$-$	$+$
Sign of $(2x + 1)(x - 1)$	$+$	$-$	$+$

E.g. 4—Solving an Inequality

- From the diagram, we see that

$$(2x + 1)(x - 1) > 0$$

- for x in the interval $(-\infty, -1/2)$ or $(1, \infty)$.

  - So the solution set is the union of these two intervals:

$$(-\infty, -1/2) \cup (1, \infty)$$

E.g. 4—Solving an Inequality

- The solution is illustrated here.

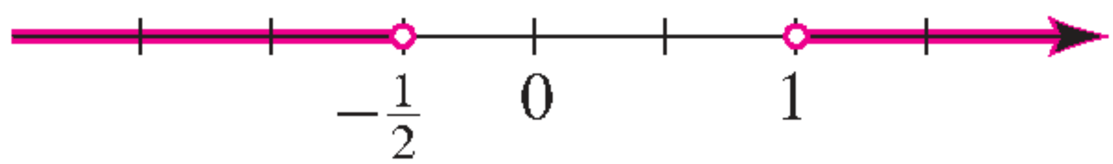


FIGURE 6

E.g. 5—Solving an Inequality with Repeated Factors

- Solve the inequality
  - $x(x - 1)^2(x - 3) < 0$

- All nonzero terms are already on one side of the inequality.
- Also, the nonzero side of the inequality is already factored.
- So we begin by finding the intervals for this inequality.

E.g. 5—Solving an Inequality with Repeated Factors

- The factors of the left-hand side are x, $(x - 1)^2$, and $x - 3$.
  - These factors are zero when $x = 0, 1, 3$.
  - These numbers divide the real line into the intervals

    $$(-\infty, 0), (0, 1), (1, 3), (3, \infty)$$

Modeling with Inequalities

Modeling with Inequalities

- Modeling real-life problems frequently leads to inequalities.
  - We are often interested in determining when one quantity is more (or less) than another.

E.g. 7—Carnival Tickets

- A carnival has two plans for tickets.

- Plan A: $5 entrance fee and 25¢ each ride
- Plan B: $2 entrance fee and 50¢ each ride

  - How many rides would you have to take for plan A to be less expensive than plan B?

E.g. 7—Carnival Tickets

- We are asked for the number of rides for which plan A is less expensive than plan B.
  - So, let:

$$x = \text{number of rides}$$

E.g. 7—Carnival Tickets

- The information in the problem may be organized as follows.

In Words	In Algebra
Number of rides	x
Cost with Plan A	$5 + 0.25x$
Cost with Plan B	$2 + 0.50x$

E.g. 7—Carnival Tickets

- Now, we set up the model.

- Cost with plan A < Cost with plan B
- $5 + 0.25x < 2 + 0.50x$
- $3 + 0.25x < 0.50x$ (Subtract 2)
- $3 < 0.25x$ (Subtract $0.25x$)
- $12 < x$ (Divide by 0.25)

  - So, if you plan to take more than 12 rides, plan A is less expensive.

E.g. 8—Fahrenheit and Celsius Scales

- The instructions on a box of film indicate that the box should be stored at a temperature between 5°C and 30°C.
  - What range of temperatures does this correspond to on the Fahrenheit scale?

E.g. 8—Fahrenheit and Celsius Scales

- The relationship between degrees Celsius (C) and degrees Fahrenheit (F) is given by:
  - $C = 5/9(F - 32)$

- Expressing the statement on the box in terms of inequalities, we have:

$$5 < C < 30$$

E.g. 8—Fahrenheit and Celsius Scales

- So, the corresponding Fahrenheit temperatures satisfy the inequalities

$$5 < \tfrac{5}{9}(F - 32) < 30$$

$$\tfrac{9}{5} \cdot 5 < F - 32 < \tfrac{9}{5} \cdot 30 \qquad \text{(Multiply by } \frac{9}{5}\text{)}$$

$$9 < F - 32 < 54 \qquad \text{(Simplify)}$$

$$9 + 32 < F < 54 + 32 \qquad \text{(Add 32)}$$

$$41 < F < 86 \qquad \text{(Simplify)}$$

E.g. 8—Fahrenheit and Celsius Scales

- Thus, the film should be stored at a temperature between 41°F and 86°F.